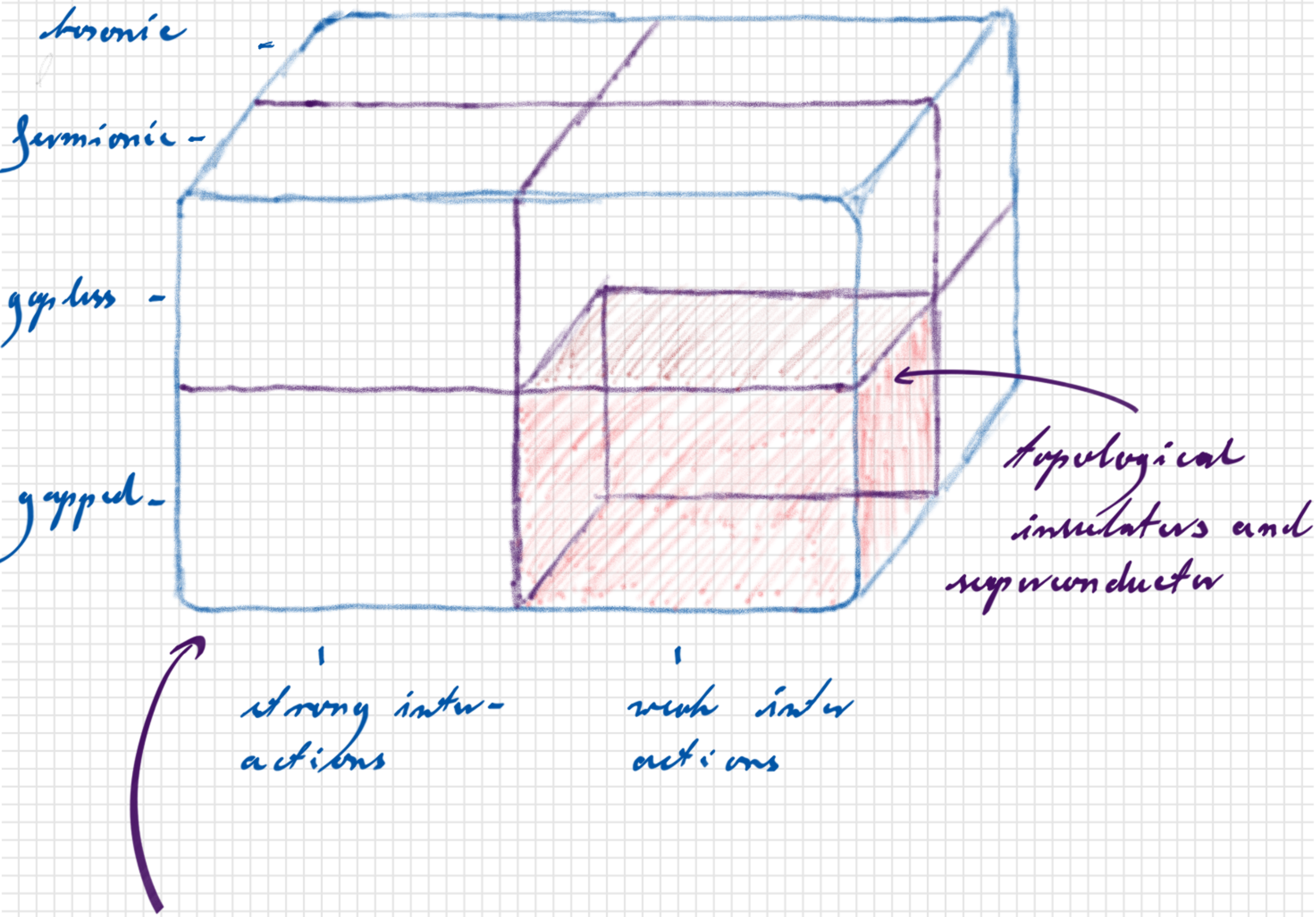
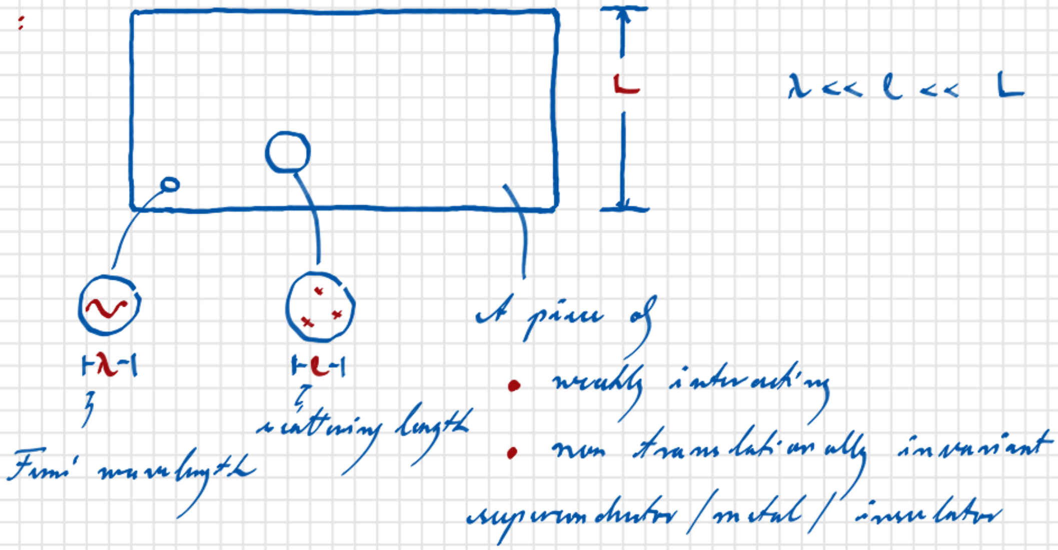


Topological Matter

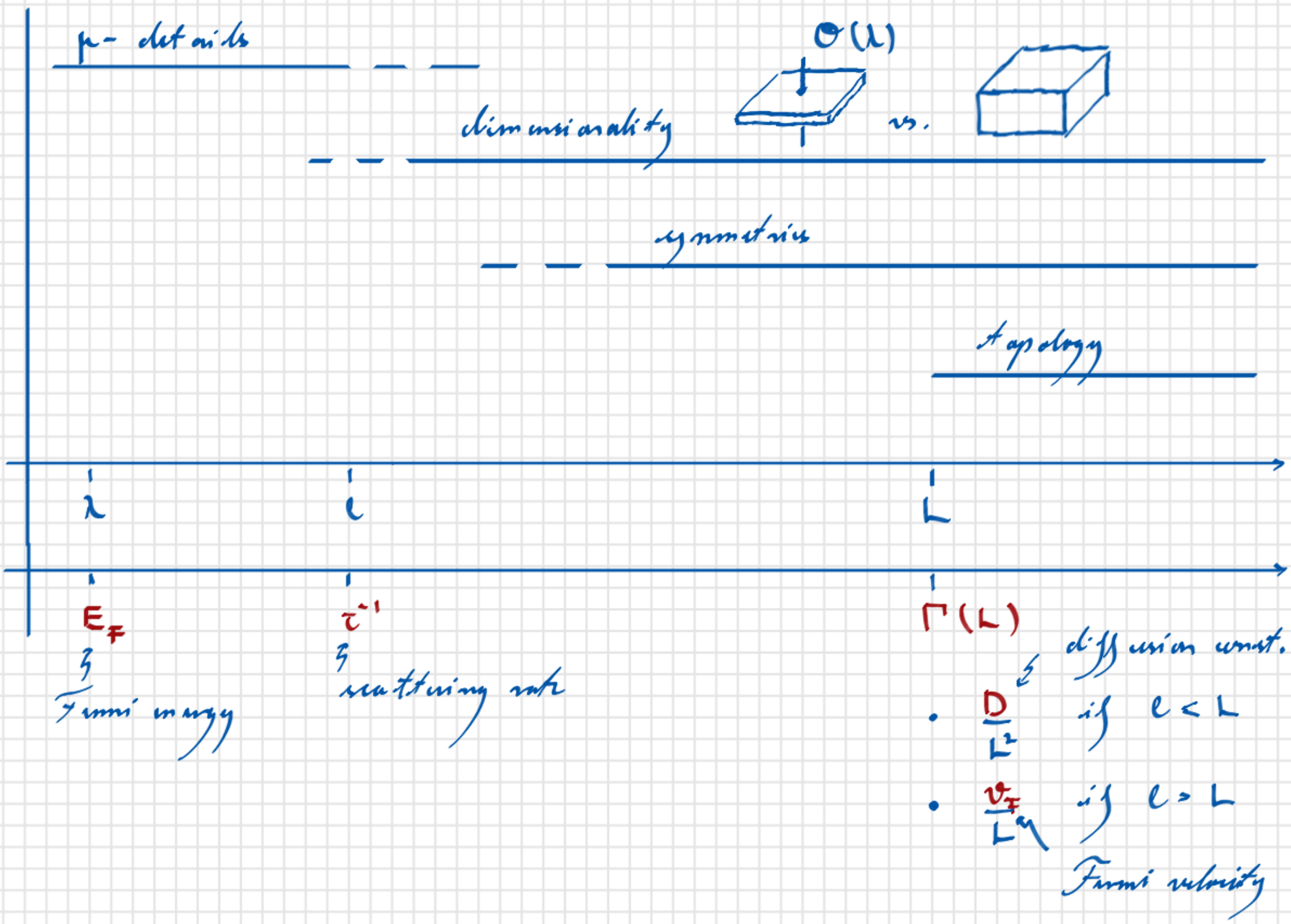
Symmetries and Topology in Non-Interacting Fermion Systems

The Setting:



$\lambda \ll l \ll L$

Physical concepts relevant to different length scales



Symmetries in QM

Given: Hilbert Space $\mathcal{H}$ of Dimension N States $|\psi\rangle$

Hamiltonian $\hat{H}$ Symmetry group $G \ni g$

Symmetry group represented on $\mathcal{H}$ through transformations $|\psi\rangle \to g|\psi\rangle$

Many symmetries of QM (Translations, rotations, crystal point operations, ...) are unitary symmetries $\equiv g \in U(N)$, the group of unitary transformations of $\mathcal{H}$.

More important to present content: anti-unitary symmetries when G is represented through anti-unitary maps $g \equiv \Theta$.

Reminder: $\Theta: \mathcal{H} \to \mathcal{H}$ is anti-unitary iff

- $\langle \Theta\psi, \Theta\psi' \rangle = \overline{\langle \psi, \psi' \rangle} = \langle \psi', \psi \rangle$
- $\Theta |z\psi\rangle = \bar{z}\, \Theta|\psi\rangle \quad z \in \mathbb{C}$

$\exists\, U \in U(N): \quad \Theta = UK \qquad K$: complex conjugation

Physics: $\Theta^2 = \pm\, \mathrm{id}$ e.g.: $K^2 = \mathrm{id}$, $\left[\begin{pmatrix} & -i \\ i & \end{pmatrix} K\right]^2 = -\mathrm{id}$.

Examples: Time reversal, particle-hole symmetry, charge conjugation symmetry, ...

Warning: Highly confusing denotations / definitions in circulation!

10 Symmetry classes (nutshell intro)

Def.: A system is *time reversal symmetric* if $\exists\, \Theta_T : \Theta_T^{-1} \hat{H} \Theta_T = +\hat{H}$

$\rightsquigarrow$ 3 possibilities:

sym.	Θ_T^2	T
$-$	$\times$	0
$+$	$+\mathrm{id}$	$+1$
$+$	$-\mathrm{id}$	-1

Comment: Often (but not always*) systems with ½-integer spin have $T=-1$

Def.: A system is *charge conjugation symmetric* iff $\exists\, \Theta_C : \Theta_C^{-1} \hat{H} \Theta_C = -\hat{H}$

$\rightsquigarrow$ 3 possibilities

sym.	Θ_C^2	C
$-$	$\times$	0
$+$	$+\mathrm{id}$	$+1$
$+$	$-\mathrm{id}$	-1

Comment: Usually** requires Nambu structure: $H = \begin{pmatrix} h & \Delta \\ \bar{\Delta} & -h^T \end{pmatrix}$ symmetric under $\Theta_C = \begin{pmatrix} & -i \\ i & \end{pmatrix} K$.

Def.: A system is called *particle-hole symmetric* if it is symmetric under $S = C \circ T$ (and therefore also $T \circ C = \underbrace{T \circ C \circ T^{-1}}_{C'} \circ T$)

$\rightsquigarrow$ 4 (!) possibilities. Comments

T	C	S
± 1	± 1	1
± 1	0	0
0	± 1	0
0	0	0
0	0	1

- $(\Theta_C \cdot \Theta_T) \hat{H} \cdot (\Theta_T^{-1} \cdot \Theta_C^{-1}) = -\hat{H} \rightsquigarrow S = 1$: $\hat{H}$ anti-commutes with ~~unitary~~ unitary operator, a *chiral symmetry*
- $T=0,\ C=0$ does not fix S

In total $\underset{T}{3} \times \underset{C}{3} + 1 = 10$ *symmetry classes*

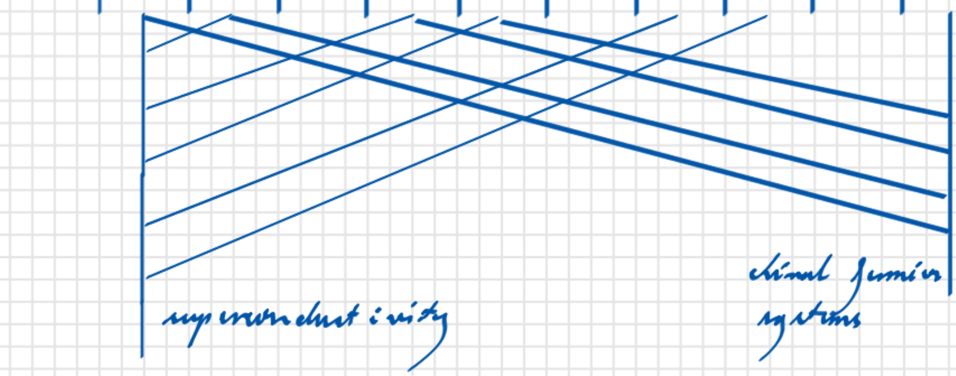
---

* Example: spinless wave functions of graphene subject to intervalley scattering have $T=-1$.

** Example: $\hat{H} = i\hat{A},\ \hat{A}^T = -\hat{A}$ C-symmetric under $\Theta_C = K$

Denotation and application areas of symmetry classes

T	+1	+1	+1	-1	-1	-1	0	0	0	0
C	+1	-1	0	+1	-1	0	+1	-1	0	0
S	1	1	0	1	1	0	0	0	0	1
label	BDI	CI	AI	DIII	CII	AII	D	C	A	AIII



Properties of symmetry classes (pre-topology)

- Systems with $C = \pm 1$ (superconductors) and AIII have spectrum symmetric around 0

$$\hat{H}|\chi\rangle = \epsilon|\chi\rangle \Rightarrow \hat{H}C|\chi\rangle = -C\hat{H}|\chi\rangle = -\epsilon C|\chi\rangle .$$

Note: For $\epsilon = 0$ $|\chi\rangle$ and $C|\chi\rangle$ can be degenerate. This is the case with topological zero modes.

- Near symmetry point $\epsilon = 0$: quantum interference phenomena

Consider, e.g., single particle density of states of gapless (!) superconductor

$$\rho(\epsilon) = -\frac{1}{2\pi}\,\mathrm{Im}\,\mathrm{tr}\left(\hat{G}^+(\epsilon)\tau_3\right) = -\frac{1}{2\pi}\,\mathrm{Im}\int dx\,\underbrace{\mathrm{tr}}_{ph}\langle x|\hat{G}^+(\epsilon)\,\tau_3|x\rangle$$

$$\left(\hat{G}^+(\epsilon)\right)^{-1} = \epsilon^+ \mathbb{1} - \underbrace{\begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}}_{\hat{H}} \qquad \Delta = -\Delta^T$$

- Gaplessness: Δ vanishes 'on average', e.g. in d,p-wave superconductor, SN hybrid systems, ...

- Symmetry: $\tau_x \hat{H}^T \tau_x = -H$. With $C = \tau_x K$: $C = +1$, $T = 0$ class D write $\hat{G}^+$ as:

'normal' retarded propagation of quasi-particle at energy ε

$$\hat{G}^+(\varepsilon) = \begin{pmatrix} \varepsilon + i0 - h & \Delta \\ \Delta^+ & -(-\varepsilon - i0 - h^T) \end{pmatrix}^{-1}$$

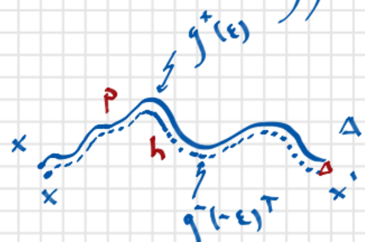
scattering between particle and hole

propagation of hole at energy $-\varepsilon$ backwards in time (h^T).

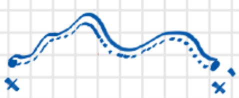
Heuristic interpretation of scattering

- Def.: $g^\pm(\varepsilon) = (\varepsilon^\pm - h)^{-1}$
- Consider $G^+_{12}(x,x,\varepsilon)$ = amplitude for particle at energy ε to get scattered into hole at energy $-\varepsilon$

Pictorially:



Note: $\langle x' | g^-(-\varepsilon)^T | x \rangle = \overline{\langle x | g^+(\varepsilon) | x' \rangle}$: hole amplitude is complex conjugate of particle amplitude



= Fourier transform of probability to propagate $x \xrightarrow{t} x'$ in time $\equiv \Pi(x,x',\varepsilon)$

time

- The dependence of Π coordinates/energy depends on type of system (disorder, size,)

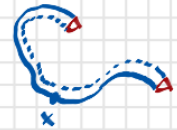
For simplicity: consider finite size system at energies $\varepsilon < \Gamma(L)$ corresponding to times $t > \Gamma(L)^{-1}$: ergodic regime $\sim$ $\Pi(x,x',t) \approx L^{-d}\,\underset{\text{Heaviside}}{\Theta(t)}$, independent of x,x' and normalized $\int dx'\,\Pi(x,x',t) = 1$. Fourier trafo:

$$\Pi(x,x',\varepsilon) = L^{-d}\,\frac{1}{\varepsilon^+}$$

Return amplitude singular at low energies. Observable consequences: band center anomalies in density of states and transport coefficients.

Example: $\rho(\epsilon)$ of class D superconductor (relevant to observation of Majorana fermions c.f. 1206.0434 for the full story). $\rho(\epsilon) = -\frac{1}{2\pi}\int dx\, \mathrm{Im}\left(\hat{G}^+_{11}(x,x,\epsilon) - \hat{G}^+_{22}(x,x,\epsilon)\right)$

Lowest order pert. theory in Δ:

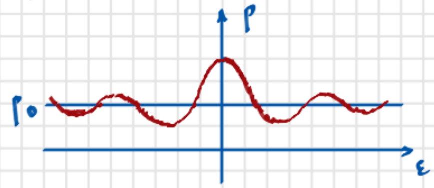
$$G^+_{11}(x,x,\epsilon) \sim$$ 

$$\sim \frac{1}{\epsilon^2}$$

Full resummation of series (complicated):

$$\rho(\epsilon) = \rho_0\left(1+\frac{\sin(\pi\epsilon/\delta)}{\pi\epsilon/\delta}\right)$$

$\delta = \rho_0^{-1}$ single particle level spacing



Remarks:
- Effect conceptually related to weak localization of ordinary metals
- Spectral peak looks like a zero-energy 'state'.
- Can easily be confused with topological zero mode (Majorana fermion)
- Not of topological origin.

Symmetries & Topology

The geometry of symmetry classes

Consider time evolution $\hat{U} = \exp(i\hat{H}t)$ generated by $\hat{H}$ of given symmetry. What is $\hat{U}$ geometrically? Example: • $(\hat{T}, \hat{C}, \hat{S}) = (0,0,0)$, $\hat{H} = \hat{H}^\dagger \rightsquigarrow \hat{U} \in U(n)$ (class A).

• $(\hat{T}, \hat{C}, \hat{S}) = (1,0,0)$, $\hat{H} = \hat{H}^\dagger$, $\hat{H} = \hat{H}^T \rightsquigarrow$

$$\hat{U} = \hat{U}^T \rightsquigarrow \hat{U} \in U(n)/\underbrace{O(n)}_{\text{antisymmetric matrices}} \quad \text{(class AI)}$$

T	+1	+1	+1	-1	-1	-1	0	0	0	0
C	+1	-1	0	+1	-1	0	+1	-1	0	0
S	1	1	0	1	1	0	0	0	0	1
label	BDI	CI	AI	DIII	CII	AII	D	C	A	AIII

BDI: $O(2n)/O(n)\times O(n)$

CI: $Sp(2n)/U(n)$

AI: $U(n)/O(n)$

DIII: $O(2n)/U(n)$

CII: $Sp(2n)/Sp(n)\times Sp(n)$

AII: $U(2n)/Sp(n)$

D: $SO(n)$

C: $Sp(n)$

A: $U(n)$

AIII: $U(2n)/U(n)\times U(n)$

- Time evolutions take values in compact symmetric spaces of rank ~ n.

  Symmetric spaces: • have geometry that looks 'the same' everywhere. Example: $U(2)/U(1)\times U(1) = A\text{III}_2$ the 2-sphere. Heuristics:

  ergodic time evolution knows about symmetry, 'uniform' otherwise

  • Have been labeled by Cartan as above

  • There are just 10 families and there are 1-1 to quantum symmetries

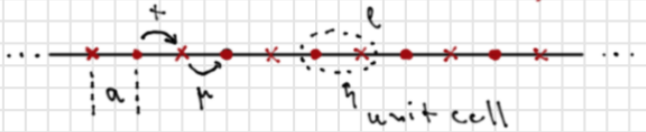
- Symmetric spaces: Rare structure to describe topology $\rightsquigarrow$

Digression before continuing to topology:

2 examples of topological insulators –

- Su – Schrieffer – Heeger (SSH) chain
- anomalous quantum Hall effect

The Su-Schrieffer-Heeger (SSH) chain (1979)



(periodic b.c.)

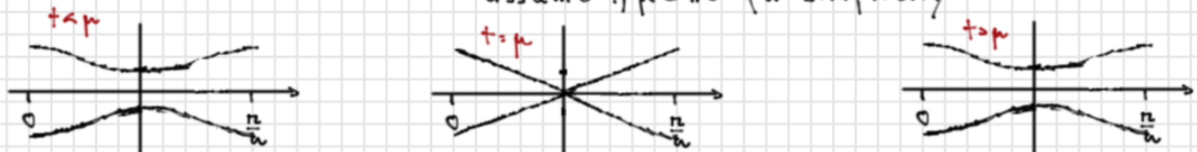
Diagonalize problem in terms of 2-component Bloch wave functions

- $\psi_k(l) = \begin{pmatrix} \psi_\times \\ \psi_\bullet \end{pmatrix}_k e^{ikl}$ $\quad k = 0, \frac{2\pi}{L}, \dots, \frac{\pi}{a}$ $\quad L = N(2a) =$ chain length

- $\hat{H}_k = \begin{array}{c} \\ \end{array}\begin{pmatrix} & q_k \\ \bar{q}_k & \end{pmatrix}\begin{array}{c} \times \\ \bullet \end{array}$ $\qquad q_k = t - e^{ik\cdot 2a}\mu$

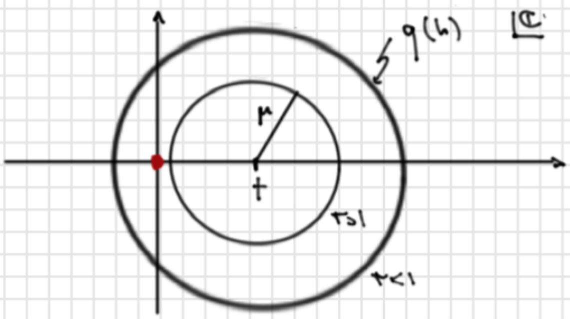
- Eigenvalues: $\varepsilon_k^{\pm} = \pm |q_k| = \left(t^2 + \mu^2 - 2t\mu\cos(2ka)\right)^{1/2}$

assume $t, \mu \in \mathbb{R}$ for simplicity



insulator $\qquad$ (semi)metal $\qquad$ insulator

- interpretation: $r \equiv \frac{t}{\mu} = 1$ marks quantum phase transition between two distinct insulating phases. No change in symmetry, no local order parameter.
- topological order: ground states of $r<1$ and $r>1$ carry distinct topological invariant

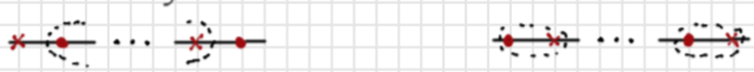


- curve $S^1 \to \mathbb{C}\setminus\{0\}$

  $k \mapsto q(k)$

  homotopically trivial/non-trivial for $r>1$ / $r<1$

- observable difference: cut system open depending on



the system with $r>1$ has/has not two zero energy boundary states sharply ($\mathcal{O}(a)$) localized at system boundaries. The insulating $r>1$ phase has a 'conducting surface'.

- Summary: $\exists$ (second order) topological quantum phase transitions without symmetry changes/local order parameter.

Anomalous QH insulator

Class A in $d=2$. Q: Can there be a QH-effect without magnetic field?

A: (Haldane) Consider lattice Hamiltonian

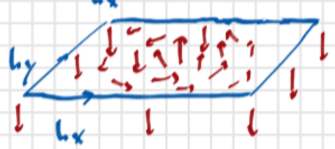
$$\hat{H} = \sin k_x\, \sigma_x + \sin k_y\, \sigma_y + (r + \cos k_x + \cos k_y)\,\sigma_z \equiv v_k \cdot \sigma = U_k \sigma_z U_k^{-1} \quad U_k \in SU(2)$$

Eigenvalues: $\varepsilon_k^{\pm} = \pm \left(\sin^2 k_x + \sin^2 k_y + (r + \cos k_x + \cos k_y)^2 \right)^{1/2}$ has gap around $\varepsilon = 0$ for $r \neq -2, 0, 2$. Eigenstates: $\psi_{\pm k} = U_k \,|\pm \tfrac{1}{2}\rangle$. Geometrically: $|\psi_{\pm k}\rangle =$ = unit vector e_z rotated into direction of $\binom{1}{0}, \binom{0}{1}$ v_k: a point on the Bloch sphere S^2. Ground state is map: $\psi_{-k}: T^2 \to S^2$, $k \mapsto \psi_{-k}$, $T^2 = S^1 \times S^1 = \{(k_x, k_y) \in [-\pi,\pi] \times [-\pi,\pi]\}$

For

$r \gg 2$:

$|r| \ll 2$:

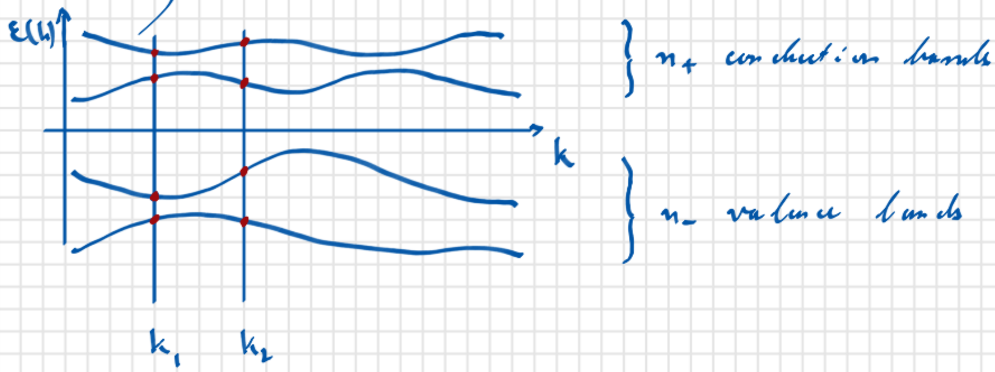


(should resemble a Hedgehog)

$\leadsto$ For $r > 2$, $r \in [0,2]$, $r \in [-2,0]$, $r < -2$, ψ_{-k} has winding $0, 1, -1, 0$, resp.

Classification of topological matter I: Homotopy Theory

Schematic of insulator band structure:



Topological information encoded in evolution of ground state. For each $k \in T^d$ – d-torus of Brillouin zone, $\hat{H}_k$ diagonalized by unitary $U_k \in U(n_+ + n_-)$ (which must be compatible with symmetries). Re-ordering of occupied / unoccupied states inessential $\rightsquigarrow$ relevant information on GS encoded in elements of Grassmannian $U(n_+ + n_-)/U(n_+)\times U(n_-)$ or symmetry restricted subset thereof.

- Example: QAH: A, $n_+ = n_- = 1$ $\quad U(2)/U(1)\times U(1) = S^2$ two sphere

  SSH: AIII, $n_+ = n_- = 1$ $\quad \hat{H}_k = \begin{pmatrix} & q_k \\ \bar{q}_k & \end{pmatrix}$ $\quad \chi_{-,k} = \begin{pmatrix} 1 \\ \underbrace{-\bar{q}_k/|q_k|}_{\in S^1} \end{pmatrix}$

  $\rightsquigarrow$ Grassmannian $\simeq S^1$

Homotopic invariants of maps $\phi: T^d \to Gr(assmanian)$ can be obtained as winding numbers or Chern numbers, or 'Chern-parities ($\mathbb{Z}_2$-insulators)'

QAH: $$W = \frac{1}{4\pi}\int_{T^2} dk_x\, dk_y\, \hat{v}_k\cdot(\partial_{k_x}\hat{v}_k \times \partial_{k_y}\hat{v}_k)$$

SSH: $$W = \frac{i}{2\pi}\int_{T^1} q_k^{-1}\partial_k q_k$$

Homotopy approach,

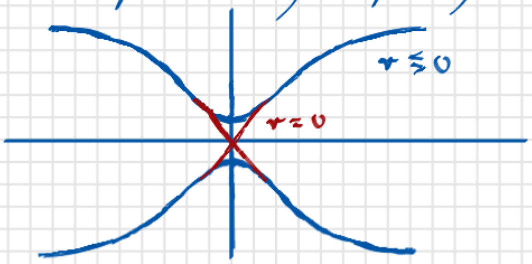
- great for classifying topologies in dependence on dimensionality / symmetry (*)
- However also abstract and
- ill suited to description of bulk boundary correspondence

(*) The systematics of maps $T^d \to G_r$ can be understood in terms of category (K-) theory. The result is the periodic table of topological insulators which describes all non-trivial homotopies (in the limit $n_\pm \to \infty$).

Cartan	0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	…
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	…
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

Classification of topological matter II: Dirac-Hamiltonian approach.

Consider spectrum of topological insulator with topological phase transition point driven by parameter r.



→ appearance of 1st order 0s in Brillouin zone.

Can be described by effective Dirac-Hamiltonians ⟺ linearization of band structures around 0s.

Example SSH chain:

$$\hat H \underset{\substack{p=1,\ t=r \\ a=1}}{=} \begin{pmatrix} & \overbrace{r-e^{ik}}^{q} \\ r-e^{-ik} & \end{pmatrix} = (r-\cos k)\,\sigma_1 + i\sin k\,\sigma_2 \simeq$$

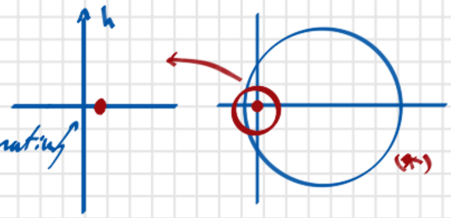
$$\underset{r=1+m}{\simeq} m\,\sigma_1 + ik\,\sigma_2$$

Topological invariant:

$$W = \frac{1}{2\pi i}\int_k q^{-1}\partial_k q = \frac{1}{4\pi i}\int_k \mathrm{tr}\left(\hat H^{-1}\partial_k \hat H\,\sigma_3\right) =$$

$$= \frac{1}{4\pi i}\int_k \mathrm{tr}\left(\left(-m\sigma_1 - ik\sigma_2\right) i\sigma_2\cdot\sigma_3\right)\frac{1}{m^2+k^2} = \frac{1}{4\pi i}\int_{-\infty}^{\infty} dk\,\frac{m}{m^2+k^2} =$$

$$= \frac{1}{2}\,\mathrm{sgn}\, m$$



- Interpretation (valid beyond SSH example)

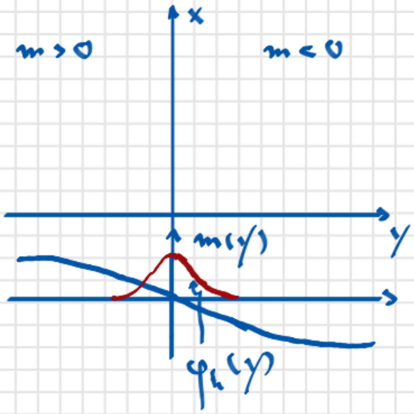
I: Linearized integration does not capture full winding number. Missing contribution from (*). However jump $W(r>0) - W(r<0)$ is predicted correctly.

II: Integrals can be UV-problematic. However always properly regularized by band structure ($H \to \left(m + \frac{k^2}{2}\right)\sigma_1 + ik\sigma_2$ in $O(k^2)$).

Dirac Hamiltonians and boundary physics.

Topological bulk ~ 0 energy boundary states ($d=1$), or gapless boundary modes ($d>1$)

Example $d=2$ QAH: $\hat{H} = \sin k_x \sigma_x + \sin k_y \sigma_y + (r - \cos k_x - \cos k_y)\sigma_z \underset{r = 2+m}{\approx}$



$$\approx k_p \sigma_p + m \sigma_z$$

$$\approx k_x \sigma_z + k_y \sigma_y + m \sigma_x$$

$\sigma_x \to \sigma_z,\ \sigma_y \to \sigma_x,\ \sigma_z \to \sigma_y$

Ansatz for wave functions: $\psi_k(x,y) = \varphi_k(y)\, e^{i k_x x}$

$$\begin{pmatrix} k_x & -i\partial_y - i m \\ -i\partial_y + i m & -k_x \end{pmatrix} \begin{pmatrix} \varphi_1(y) \\ \varphi_2(y) \end{pmatrix} = k_x \begin{pmatrix} \varphi_1(y) \\ \varphi_2(y) \end{pmatrix}$$

has solutions concentrated at $y=0$ boundary and linearly dispersive in x-direction (e.g. for $m(y) = -cy$, $\rightsquigarrow (\varphi_1(y) \sim e^{-\frac{c}{2}y^2}, \varphi_2(y) = 0)$). Existence of solutions guaranteed by index theorems.

III: Dirac approach well suited to describe boundary modes

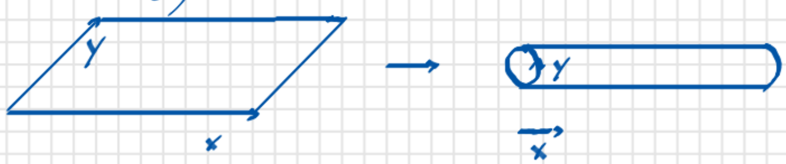
Dirac approach and periodic table.

Periodic table shows (Bott) periodic diagonal structures. E.g.:

	1	2	3	4
AIII	$\mathbb{Z}$		$\mathbb{Z}$	
A		$\mathbb{Z}$		$\mathbb{Z}$

Can be understood from Dirac approach by dimensional reduction

Example: Compactify $d=2$ QAH insulator (class A) in one direction to a cylinder



$$\hat{H}_2 = k_x \sigma_x + \sin k_y \sigma_y + (1 + m - \cos k_y)\sigma_z \xrightarrow[\text{via quantization}]{\S} k_x \sigma_x + m \sigma_z \equiv \hat{H}_1 \qquad [\hat{H}_1, \sigma_y]_+ = 0$$

$\rightsquigarrow$ $\hat{H}_1$ is in AIII. Can show: non-vanishing Chern number of $\hat{H}_2$ $\rightsquigarrow$ Non-vanishing winding number of $\hat{H}_1$

IV: Dirac insulators great for understanding dimensional structures and topology.

Classification of topological matter III: Beyond translational invariance

So far approaches emphasized momentum quantum numbers: translational invariance required.

Q: How describe topology in aperiodic structures (disorder, incommensurate potentials, irregular geometries)

Real space approaches to topology:
- scattering theory (cf. 1101.1749)
- gauge theory

topological

monitor scattering matrix eigenvalues.

Gauge theory approach.

Example: Consider multi-channel class D superconductor wires

$$\hat H' = -\tau_x \hat H'^T \tau_x$$

particle hole

apply similarity transformation: $\hat H = e^{\frac{i\pi}{4}\tau_1} \hat H' e^{-\frac{i\pi}{4}\tau_1}$

physically: The Majorana basis: $(c, c^+) \to (\gamma \pm i\nu)\frac{1}{\sqrt{2}}$

$$\hat H^T = -\hat H$$ an anti-symmetric purely imaginary matrix.

Consider bilinear form: $\gamma^T H \gamma$. Q.M. gauge symmetry $c \to e^{i\varphi} c$ $\varphi \in [0, 2\pi]$ broken to $\varphi \in \{0, \pi\}$. Gauge group $\mathbb{Z}_2$. Majorana fermions

Consider ring shaped quantum wire

$H_{ij} = -H_{ji}$

- Send $\mathbb{Z}_2$ gauge flux, φ
- Represent it by vector potential A: $\int_{\text{ring}} A = \varphi$.
- Deform A such that $A_{ij} = 0$ $i \neq 1, j = N$
- Flux $\varphi = \pi$ amounts to sign inversion: $H_{1N} \to -H_{1N}$
- Consider adiabatic change

$H_{ij} \to \overset{\pi}{s}\, H_{ij}$

$s \in [-1, 1]$

0 π 0≅2π

$s = 1 \ldots, -1, \ldots, 1$

- $s = 1 \to s = -1$ must cross $s = 0$, the system is cut open along the adiabatic evolution.
- Iff topological:

Adiabatic evolution must lead to emergence of 0-modes.

$s = 0$: emergence of zero energy Majorana states.